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TESTING FOR EXPONENTIALITY AGAINST IFRA ALTERNATIVES USING A U-STATISTIC PROCESS

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Summary

This article presents the study of a U–statistic process arising in the problem of testing exponentiality versus nonexponential increasing failure rate average (IFRA) distributions. Weak convergence of this U-statistic process to a Gaussian process is proved and a functional of this process is proposed as the test statistic. It is shown that this test statistic has desirable asymptotic properties and also a higher asymptotic relative efficiency compared to some other (existing) tests in the literature. Results of a Monte Carlo study carried out to obtain power estimates for small samples are presented.

Key words: Exponentiality tests; incre ing failure rate average; U-statistics.

1. Introduction

Among the various nonparametric classes of life distributions, reflecting specific ageing properties, the class of increasing failure rate average (IFRA) distributions has gained considerable importance. It is the smallest class of life distributions containing the exponential distribution which is closed under the formation of coherent systems and its elements also describe life lengths experiencing damage from random shocks under fairly general assumptions (see Barlow and Proschan, 1975). This class is definid as follows.

Definition 1.1: A life distribution F belongs to the IFRA class if and only if

$$\overline{F}(bx) \geq [\overline{F}(x)]^b, \text{ for all } 0 < b < 1, \quad x > 0.$$

Let $X_1, \cdots, X_n$ be a random sample of size n from a continuos IFRA life distribution function F. Tests for the exponentiality (H_0) versus the nonexponentiality (H_1) of F have been proposed by several authors (see, for example, Barlow and Campo, 1975, Bergman, 1977 and Klefsjo, 1983). Using Definition 1.1 of IFRA life distributions, and by considering $\delta_F = \int_0^1 \int_0^\infty \frac{(1+b)}{4} \overline{F}(bx) dF(x) db$ as a measure of deviation of F from exponentiality towards IFRA alternatives, Ahmad(1980) proposed a test statistic

$$U_n = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \frac{(1+b)}{4} I(X_i > bX_j),$$

where $I(A)$ denotes the indicador function of the set A. Deshpande(1983) proposed a class of statistics $\{J_n(b) : 0 < b < 1\}$, where

$$J_n(b) = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} I(X_i > bX_j),$$

based on the parameter

$$M_F(b) = \int_0^\infty \overline{F}(bx) dF(x).$$

Since $M_F(b) = \frac{1}{b+1}$ under H_0 and $M_F(b) > \frac{1}{b+1}$ under H_1, viewing $M_F(b)$ as a measure of devation of F from exponentiality towards the IFRA alternatives, we reject H_0 in favor of H_1 for large values of $J_n(b)$. Using Hoeffding's(1948) results on U-statistics, both Ahmad(1980) and Deshpande(1983) established the asymptotic normality of U_n and $J_n(b)$ respectively and computed the asymptotic relative efficiencies (ARE's) of their tests relative to the tests proposed by Hollander and Proschan(1972) and Bickel and Doksum(1969). The choice of optimal values of b in $J_n(b)$ is discussed in Tiwari, Jammalamadaka and Zalkikar(1989) and in Bandyopadhyay and Basu(1989).

In this paper, we look at $\{J_n(b); 0 \leq b \leq 1\}$ as a U-statistic process in b, rather than as a set of test statistics $J_n(b)$ for each b. U-statistic process in a very general setup has been discussed by Noland and Pollard(1987), but their interest has no direct bearing on this application. In Section 2 we study the weak convergence of this process. In Section 3 we use the results of Section 2, to derive the asymptotic distribution of a test statistic that is independent of b, for testing the exponentiality of F. Asymptotic relative efficiency (ARE) calculations and Monte Carlo power estimates of the proposed test for small samples are presented in Section 4.

2. The $J_n(b)$ process

Note that, for $b \in [0,1]$,

$$J_n(b) = \frac{1}{2} + \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} I\left(\frac{X_{(i)}}{X_{(j)}} > b\right), \tag{2.1}$$

where $X_{(1)} < \cdots < X_{(n)}$ are the order statistics of the random sample $X_1, \cdots, X_n$. Since F is continuous, there are no ties with probability 1 and it is clear from (2.1) that the knowledge of a sample path of the $J_n(b)$ process is equivalent to knowing the $n(n-1)/2$ ratios of the observations $X_1, \cdots, X_n$ which are less than 1. The remaining $n(n-1)/2$ ratios bigger than 1 can be obtained by taking reciprocals, and exactly n ratios are equal to 1. Hence $\{J_n(b): 0 \leq b \leq 1\}$ as a process on $D[0,1]$, the space of functions on $[0,1]$ that are right continuos and have left hand limits, contains all the information about the ratios $(\frac{X_{(1)}}{X_{(2)}}, \frac{X_{(2)}}{X_{(3)}}, \cdots, \frac{X_{(n-1)}}{X_{(n)}})$, a maximal invariant statistic under the groups of scale transformations. Consequently the asymptotic distribution of the statistic proposed in Section 3 as well as that of any other scale invariant test statistic can be obtained from the weak convergence of the $J_n(b)$ process as these are functionals of this process. However, for some of these test statistics it may be more convenient to use other methods. The weak convergence of the $J_n(b)$ process proved in Appendix A is stated in the following theorem.

Theorem 2.1. *The sequence of processes* $\{n^{1/2}(J_n(b) - M_F(b)): 0 \leq b \leq 1\}$ *converges weakly to a Gaussian process with mean* 0 *and covariance kernel given by*

$$K(b_1, b_2) = \begin{cases} \int_0^{\infty}[F(\frac{x}{b_1}) + \overline{F}(b_1x)][F(\frac{x}{b_2}) + \overline{F}(b_2x)]dF(x) - 4M_F(b_1)M_F(b_2), & 0 \leq b_1, b_2 \leq 1 \\ 0 & \text{o.w.} \end{cases} \tag{2.2}$$

3. One sample IFRA test

Given a random sample $X_1, \cdots, X_n$ from a continuos IFRA life distribution F, we develop a test procedure for testing

$$H_0 : \overline{F}(bx) = [\overline{F}(x)]^b \text{ for all } x \geq 0 \text{ and for all } 0 \leq b \leq 1,$$

versus

$$H_1 : \overline{F}(bx) \geq [\overline{F}(x)]^b \text{ for all } x \geq 0 \text{ and for all } 0 < b < 1,$$

with strict inequality for some x. To measure the deviation of F from H_0 towards H_1, consider the parameter (cf. Zalkikar, 1988)

$$\begin{aligned} \Delta(F) &= \int_0^1 \int_0^{\infty} \overline{F}(bx)dF(x)db \\ &= \int_0^1 M_F(b)db \end{aligned} \tag{3.1}$$

which under H_0 has the value $\ell n2$ and is larger than $\ell n2$ under H_1. Substituting the empirical distribution function $\widehat{F}_n$ for F and noting that $\Delta(\widehat{F}_n) = \int_0^1 M_{\widehat{F}_n}(b)db$ is asymptotically equivalent to

$$T_n = \int_0^1 J_n(b)db,$$

we propose T_n as a test statistc for testing H_0 versus H_1, and reject H_0 in favor of H_1 for large values of T_n. The asymptotic normality of T_n follows from the application of Theorem 2.1 and the continuous mapping Theorem (cf.Billingsley, 1969, p.30) and is given by

Theorem 3.1. *$\sqrt{n}(T_n - \Delta(F))$ has limiting $N(0, \sigma^2)$ distribution, where σ^2 is given by*

$$\sigma^2 = \int_0^1 \int_0^1 K(b_1, b_2)db_1 db_2 \tag{3.2}$$

and $K(b_1, b_2)$ is the covariance kernel given by (2.2).

When F is exponential (with unspecified parameter μ), it follows from (3.1) that $\Delta(F) = \int_0^1 \frac{1}{b+1}db = \ell n2$ and from (2.2) that

$$K(b_1, b_2) = 1 - \frac{2(1 + b_1 b_2)}{(b_1 + 1)(b_2 + 1)} + \frac{b_1 b_2}{b_1 b_2 + b_1 + b_2} - \frac{b_1}{b_1 b_2 + b_1 + 1}$$
$$- \frac{b_2}{b_1 b_2 + b_2 + 1} + \frac{1}{b_1 + b_2 + 1}, 0 \leq b_1, b_2 \leq 1. \tag{3.3}$$

Substituting (3.3) in (3.2) we get $\sigma^2 = 0.012$ and we have the following corollary.

Corollary 3.2. *Under the null hypothesis of exponentiality $\sqrt{n}(T_n - \ell n2)$ has the $N(0, 0.012)$ limiting distribution.*

It follows from Corollary 3.2 that the sequence of test $\{T_n\}$ is consistent against all continuous (nonexponential) IFRA alternatives.

The computational form of the test statistic T_n is

$$T_n = \frac{1}{n(n-1)} \sum_{1 \le i \ne j \le n} \min(1, \frac{X_i}{X_j}) \tag{3.4}$$

In terms of the order statistics $X_{(1)}, \cdots, X_{(n)}$ of the random sample $X_1, \cdots, X_n$,

$$T_n = \frac{1}{2} + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{X_{(i)}}{X_{(j)}}$$

Remark 3.1: From (3.4) note that T_n is a U-statistic with kernel

$$h(x_1, x_2) = \min\,(1, \frac{x_1}{x_2}), x_1, x_2 > 0 \tag{3.5}$$

for an estimable parameter $P(U < X/Y)$ where U, X and Y are nonnegative, independent r.v.'s with X and Y having the same distribution function F, and U is the uniform r.v. on $[0,1]$. Therefore, one can use the theory of U-statistics to give an alternative proof of Theorem 3.1. This is given in Appendix B. However, it is clear that not every functional, $\Psi(J_n(b))$, of the U-statistic process $J_n(b)$ can be written as a U-statistic. Thus, the simple alternative proof we have provided in Appendix B for T_n cannot be carried through, for example, when dealing with $\sup_b J_n(b)$.

4. Efficiency and power computations

Let $\{F_{\theta_n}\}$ be a sequence of alternatives with $\theta_n = \theta_0 + \frac{a}{\sqrt{n}}$, where a is an arbitrary positive constant and F_{θ_n} is exponential with scale parameter 1. The extended U-statistics theorem ensures that the standard regularity conditions in Noether's(1955) theorem (cf. Randles and Wolfe, 1979, p. 147) are satisfied for T_n, U_n and $J_n(b)$.

The ARE of the T_n with respect to (w.r.t) $J_n(b)$ is given by

$$\rho_F(T_n, J_n(b)) = \left[\frac{\Delta^{(1)}(\theta_0)}{M_F^{(1)}(b;\theta_0)}\right]^2 \frac{\sigma^2(b)}{\sigma^2} \tag{4.1}$$

where σ^2 and $\sigma^2(b)$ are asymptotic variances of $n^{1/2}T_n$ and $n^{1/2}J_n(b)$ under H_0 respectively, and $\Delta^{(1)}(\theta_0)(M_F^{(1)}(b;\theta_0))$ is the derivative with respect to θ of $\Delta(\theta)(M_F(b;\theta))$, the asymptotic mean of $T_n(J_n(b))$ under F_θ, evaluated at $\theta = \theta_0$. Note that $\rho_F(T_n, J_n(b))$ is the square of the ratio of the efficacies of T_n and $J_n(b)$ tests. For the computations of ARE we consider the Weibull family of alternatives with d.f. $F_\theta(x) = 1 - e^{-x^\theta}, x > 0, \theta > 1$. Then from (4.1), $\rho_F(T_n, J_n(b))$ is given by

$$\rho_F(T_n, J_n(b)) = \left[\frac{0.129}{-b\ell nb/(b+1)^2}\right]^2 \frac{\sigma^2(b)}{0.012} \tag{4.2}$$

where

$$\sigma^2(b) = 1 + \frac{b}{b+2} + \frac{1}{2b+1} + \frac{2(1-b)}{b+1} - \frac{2b}{b^2+b+1} - \frac{4}{(b+1)^2}.$$

The efficacy of the test $J_n(0.44)$ is maximum among the test $\{J_n(b) : 0 < b < 1\}$ for Weibull alternatives (see Tiwari, Jammalamadaka and Zalkikar, 1989, or Bandyopadhyay and Basu, 1989). From (4.2), the ARE of the T_n test with respect to the $J_n(0.44)$ test is 1.0225. Similar calculations for U_n yield the ARE of T_n test with respect to the U_n test as 4.762. These ARE's indicate the higher efficiency of T_n. Small sample performance of T_n also points in the same direction, as seen from the following tables. Table 4.1 gives Monte Carlo powers of the T_n test for sample size varing from $n = 5$ to $n = 15$ for Weibull and linear failure rate alternatives. The level of significance used is $\alpha = 0.05$. Table 4.2 enables us to compare the T_n test w.r.t. $J_n(0.44)$ test in terms of power of the test. Here the alternatives are Weibull and the sample size is 15. The simulation study carried out for T_n shows that for samples of size $n \geq 10$, it is safe to use normal approximation for T_n.

Table 4.1

Monte Carlo Powers of T_n test with $\alpha = 0.05$.

n/θ	1.25	1.5	2	2.5
5	0.112	0.182	0.398	0.579
	(0.103)	(0.126)	(0.131)	(0.135)
7	0.112	0.215	0.455	0.726
	(0.163)	(0.184)	(0.185)	(0.194)
9	0.180	0.337	0.690	0.903
	(0.191)	(0.196)	(0.196)	(0.249)
11	0.180	0.337	0.767	0.939
	(0.195)	(0.202)	(0.245)	(0.268)
13	0.180	0.424	0.846	0.979
	(0.195)	(0.223)	(0.262)	(0.294)
15	0.180	0.465	0.905	0.990
	(0.206)	(0.227)	(0.262)	(0.328)

Table 4.2

Monte Carlo Powers of T_n and $J_n(0.44)$ tests with $\alpha = 0.05, n = 15$ and Weibull alternatives

Test/θ	1.25	1.5	2	2.5
T_n	0.180	0.465	0.905	0.990
$J_n(0.44)$	0.180	0.435	0.865	0.982

In Table 4.1, the values without brackets correspond to Weibull alternatives with $F_\theta(x) = 1 - \exp(-x^\theta), x > 0, \theta > 1$ and the values in brackets correspond to Linear Failure rate alternative with $F_\theta(x) = 1 - \exp(-x - \frac{\theta}{2}x^2), x > 0, \theta > 0$.

5. Discussion

The U-statistic process considered in this paper contains all the information about ratios of the observations in the sample and provides a test statistic for testing exponentiality against IFRA alternatives. This test statistic has limiting normal distribution and the simulation results show that a sample of size 10 or more is adequate for the use of asymptotic results. The usefulness of this new test procedure lies in the fact that while using this procedure one does not face the problem of choosing a test from the class of tests as is the case with Deshpande's tests, and the test performs reasonably well in terms of ARE and power.

Appendix A

The proof of Theorem 2.1 is given through the following lemmas.

Lemma A.1. *Let $J_n^*(b) := J_n(b) - M_F(b)$. For any fixed $b_1, \cdots, b_l \in [0,1]$ the (finite dimensional) joint distribution of $\{n^{1/2} J_n^*(b_i), i = 1, 2, \cdots, l\}$ converges to l-variate normal distribution, where the variance matrix, Σ, is given by (A.2).*

Proof: For any fixed $a_1, \cdots, a_l$ it is sufficient to show that $n^{1/2} \sum_{i=1}^{l} a_i J_n^*(b_i)$ is asymptotically normal. For this we use the projection of $J_n^*(b)$ on the class of sums of i.i.d r.v.'s given by

$$V_n(b) = \frac{2}{n}\sum_{j=1}^{n}\left[\frac{1}{2}(F(\frac{X_j}{b}) + \overline{F}(bX_j) - M_F(b))\right]$$

$$= \sum_{j=1}^{n} U_b(X_j),$$

say. Note that (cf. Randles and Wolf, 1979, p.83)

$$nE\left[\sum_{i=1}^{l} a_i J_n^*(b_i) - \sum_{i=1}^{l} a_i V_n(b_i)\right]^2$$

$$\leq nlE\left[\sum_{i=1}^{l} a_i^2 (J_n^*(b_i) - V_n(b_i))^2\right] \to 0 \text{ as } n \to \infty \tag{A.1}$$

Since $Y_j := \sum_{i=1}^{l} a_i U_{b_i}(X_j), j = 1, \cdots, n$ are i.i.d r.v.'s with mean 0 and finite variance, the Lindberg-Levy version of the central limit theorem gives the asymptotic normality of $n^{1/2}\sum_{j=1}^{n} Y_j = n^{1/2}\sum_{i}^{l} = a_i V_n(b_i)$ and hence from (A.1) that of $n^{1/2}\sum_{i=1}^{l} a_i J_n^*(b_i)$. It is easy to verify that the variance-covariance matrix of $n^{1/2} J_n^*(b_i), i = 1, \cdots, l$ is

$$\Sigma = ((K(b_1, b_j))), \tag{A.2}$$

where $K(b_i, b_j)$ is as defined in (2.2). □

Lemma A.2. *The family of probability measures induced on $D[0,1]$ by the processes $\{n^{1/2} J_n^*(b); 0 \leq b < 1\}$ is tight.*

Proof: Since $\overline{M}_F(b) = 1 - M_F(b)$ is a nondecreasing continuous function of $b \in [0,1]$, by Theorem 15.6 of Billingsley (1968, p. 128) it is sufficient to show that

$$nE\left(|J_n^*(b_3) - J_n^*(b_2)||J_n^*(b_2) - J_n^*(b_1)|\right) \leq \left(\overline{M}_F(b_3) - \overline{M}_F(b_1)\right)^{2\alpha},$$

for $b_1 \leq b_2 \leq b_3$, where $\alpha > \frac{1}{2}$. By Cauchy-Schwartz inequality

$$\begin{aligned} &nE(|J_n^*(b_3) - J_n^*(b_2)||J_n^*(b_2) - J_n^*(b_1)|) \\ &\leq n\{E\left(J_n^*(b_3) - J_n^*(b_2)\right)^2 E(J_n^*(b_2) - J_n^*(b_1))^2\}^{1/2} \end{aligned} \tag{A.3}$$

Since $J_n(b_3) - J_n(b_2)$ is a U-statistic, by a standard result from the theory of U-statistic (cf. Denker, 1985)

$$E\left(J_n^*(b_3) - J_n^*(b_2)\right)^2 \leq \frac{2}{n}\left(\overline{M}_F(b_3) - \overline{M}_F(b_2)\right)\left(1 - \overline{M}_F(b_3) + \overline{M}_F(b_2)\right) \tag{A.4}$$

and

$$E\left(J_n^*(b_2) - J_n^*(b_1)\right)^2 \leq \frac{2}{n}\left(\overline{M}_F(b_2) - \overline{M}_F(b_1)\right)\left(1 - \overline{M}_F(b_2) + \overline{M}_F(b_1)\right) \tag{A.5}$$

Combining (A.3), (A.4) and (A.5) yields

$$\begin{aligned} &nE\left(|J_n^*(b_3) - J_n^*(b_2)||J_n^*(b_2) - J_n^*(b_1)|\right) \\ &\leq 2\left[\left(\overline{M}_F(b_3) - \overline{M}_F(b_2)\right)\left(\overline{M}_F(b_2) - \overline{M}_F(b_1)\right)\left(1 - \overline{M}_F(b_3)\right.\right. \\ &\left.\left. + \overline{M}_F(b_2)\right)\left(1 - \overline{M}_F(b_2) + \overline{M}_F(b_1)\right)\right]^{1/2} \end{aligned} \tag{A.6}$$

Note that

$$\begin{aligned} &(1 - \overline{M}_F(b_3) + \overline{M}_F(b_2))(1 - \overline{M}_F(b_2) + \overline{M}_F(b_1)) \\ &\leq \left[\left(\overline{M}_F(b_3) - \overline{M}_F(b_1)\right)^{\delta} + (\overline{M}_F(b_3) - \overline{M}_F(b_1))^{1/2}\right]^2 \end{aligned} \tag{A.7}$$

for some $\delta > 0$. Substituting (A.7) in (A.6) and simplifying gives

$$\begin{aligned} &nE(|J_n^*(b_3) - J_n^*(b_2)||J_n^*(b_2) - J_n^*(b_1)|)^2 \\ &\leq c(\overline{M}_F(b_3) - \overline{M}_F(b_1))^{2\alpha}, \end{aligned}$$

where $c > 0$ is a constant, and $\alpha = \frac{1}{2}(1 + \min(\delta, \frac{1}{2}))$. □

Appendix B

Alternative proof of Theorem 3.1: From Hoeffding's(1948) results, the distribution of $\sqrt{n}(T_n - \Delta(F))$ is asymptotically normal with mean 0 and variance $4\ \zeta_1$, where

$$\zeta_1 = E\left(\phi_1^2(X_1)\right) - (\Delta(F))^2, \tag{B.1}$$

$\phi_1(X_1) = E[h^*(x_1, x_2)]$, and h^* is a symmetric version of h in (3.5) given by

$$h^*(x_1, x_2) = \frac{1}{2}\left[\min\left(1, \frac{x_1}{x_2}\right) + \min\left(1, \frac{x_2}{x_1}\right)\right] \tag{B.2}$$

Using the fact that $\min(1, \frac{x_1}{x_2}) = E(I(U < \frac{x_1}{x_2}))$, where U is a uniform r.v. on $[0,1]$, and (B.2) in (B.1) and simplifying gives

$$\phi_1(x_1) = \frac{1}{2}\int_0^1 \left\{F\left(\frac{x_1}{U}\right) + \overline{F}(ux_1)\right\} du.$$

and hence,

$$E\left(\phi_1(X_1)\right) = \frac{1}{2}\int_0^1\int_0^\infty \left\{F\left(\frac{x}{u}\right) + \overline{F}(ux)\right\} dF(x)du = \Delta(F) \tag{B.3}$$

$$E\left(\phi_1^2(X_1)\right) = \frac{1}{4}\left[\int_0^1\int_0^1\int_0^\infty \left\{F\left(\frac{x}{u}\right) + \overline{F}(ux)\right\}\left\{F\left(\frac{x}{u}\right) + \overline{F}(vx)\right\} dF(x)dudv\right] \tag{B.4}$$

Using (B.3)and (B.4) in (B.1) givens $4\zeta_1 = \sigma^2$ with σ^2 definid by (3.2).

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